

The Even and Odd Supersymmetric Hunter - Saxton and Liouville Equations

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Abstract

It is shown that two different supersymmetric extensions of the Harry Dym equation lead to two different negative hierarchies of the supersymmetric integrable equations. While the first one yields the known even supersymmetric Hunter - Saxton equation, the second one is a new odd supersymmetric Hunter - Saxton equation. It is further proved that these two supersymmetric extensions of the Hunter - Saxton equation are reciprocally transformed to two different supersymmetric extensions of the Liouville equation.

1 Introduction

The list of unusual behavior of the supersymmetric integrable systems is rather long. It appears that during the supersymmetrization of the classical systems, some typical supersymmetric effects, compared with the classical theory, occur. The non-uniqueness of the roots of the supersymmetric Lax operator [13], the lack of the bosonic reduction to the classical equations [9] and the occurrence of the non-local conservation laws [8, 6] have been discovered in the last century. Recently it appeared that both even and odd Hamiltonian operators could be used to the supersymmetrization of the classical integrable systems [2, 11, 12, 15]. These effects rely strongly on the descriptions of the generalized classical systems which we would like to supersymmetrize.

In this paper we prolong this list, namely we show that the Hunter - Saxton (HS) equation, similarly as the Harry Dym (HD) equation, could be supersymmetrized in two different manners. It is known that the classical HD equation is

supersymmetrized in two ways, either by even or by odd supersymmetric Hamiltonian operators [2, 11]. It should be remarked that in addition to these two cases, a super HD equation was deduced from the fermionic extension of energy-dependent Schrödinger operator [1].

The even supersymmetric HD equation is a multi - Hamiltonian system and its negative hierarchy contains the even supersymmetric generalization of the HS equation. The second Hamiltonian structure for this extension is generated by the supersymmetric centerless Virasoro algebra. In the second approach, where we use the odd Hamiltonian operators, an odd supersymmetric HD equation is obtained from the Lax representation. However this equation admits bi - Hamiltonian formulation only and does not possess the ‘first’ Hamiltonian operator.

As we shall show that, from the knowledge of the second and third odd supersymmetric Hamiltonian structures of the odd supersymmetric HD equation, it is possible to construct new negative hierarchy of equations. The flows of this hierarchy contain a new supersymmetric extension of the HS equation.

It is well known that the classical HS equation under reciprocal transformation reduces to the Liouville equation [5]. We show that it is possible to apply the supersymmetric analogue of reciprocal transformation described in [11] to two different supersymmetric extensions of the HS equation, and as a result two supersymmetric extensions of the Liouville equation are obtained.

The paper is organized as follows. In the first section we recapitulate the known facts on the bi - Hamiltonian formulation of the classical HS equation and explain its connections with the negative hierarchy of the HD equation. In the next section the even hierarchy of the supersymmetric HD equation as well as its negative hierarchy, which contains the even supersymmetric extension of the HS equation, are presented. Also this section describes the reciprocal link between the even supersymmetric HS equation and a new supersymmetric Liouville equation. The third section contains our main result where the negative hierarchy of the odd supersymmetric HD equation is introduced. This hierarchy takes an odd supersymmetric HS equation as one of its flows. Similar to the previous section we describe a reciprocal link between this odd supersymmetric HS equation and a new supersymmetric Liouville equation. In the appendix a simple proof is presented for the fact that the third Hamiltonian operators of the even as well odd supersymmetric HD equations do satisfy the Jacobi identity.

2 Hunter - Saxton equation

The Hunter - Saxton equation [7]

$$u_{xxt} = 2u_x u_{xx} + u u_{xxx} \quad (1)$$

could be constructed in two different approaches. In the first approach this equation is considered as the special limit of the Camassa - Holm equation [5] while in the second one it is regarded as a member of the negative hierarchy of the Harry Dym equation.

Let us briefly recapitulate these scenarios.

Both the HS equation and the Camassa - Holm equation could be embedded into the general model

$$\lambda u_t - u_{xxt} = \frac{1}{2} \left(-3\lambda u^2 + 2uu_{xx} + u_x^2 \right)_x, \quad (2)$$

where λ is an arbitrary parameter. For $\lambda = 1$ the equation (2) is just the Camassa - Holm equation [3, 4] while for $\lambda = 0$ it becomes the HS equation (1).

The general equation (2) is a bi-Hamiltonian system

$$m_t = K_1 \frac{\delta H_1}{\delta m} = K_2 \frac{\delta H_2}{\delta m},$$

where $m = \lambda u - u_{xx}$ and

$$\begin{aligned} H_1 &= \frac{1}{2} \int dx \, um, & H_2 &= -\frac{1}{2} \int dx \, (\lambda u^3 + uu_x^2), \\ K_1 &= -(\partial_x m + m\partial_x), & K_2 &= (\lambda - \partial_x^2)\partial_x. \end{aligned}$$

To put the HS equation into the negative HD hierarchy, we first consider the bi - Hamiltonian structure of the HD equation

$$w_t = P_1 \frac{\delta H_{-1}}{\delta w} = P_2 \frac{\delta H_{-2}}{\delta w} = (w^{-\frac{1}{2}})_{xxx} \quad (3)$$

where

$$\begin{aligned} P_1 &= \partial_x^3, & P_2 &= \partial_x w + w\partial_x = 2w^{\frac{1}{2}}\partial_x w^{\frac{1}{2}}, \\ H_{-1} &= 2 \int dx \, w^{\frac{1}{2}}, & H_{-2} &= \frac{1}{8} \int dx \, w^{-\frac{5}{2}} w_x^2. \end{aligned} \quad (4)$$

As we see in both approaches we deal with the same bi - Hamiltonian structure. In fact the HD equation possesses the multi - Hamiltonian structure which is constructed out of the recursion operator $R = P_2 P_1^{-1}$ and P_2 and reads as $P_{n+2} = R^n P_2, n = 1, 2, \dots$. For example the third Hamiltonian structure is

$$w_t = P_3 \frac{\delta H_{-3}}{\delta w}$$

where

$$\begin{aligned} P_3 &= P_2 P_1^{-1} P_2 = (\partial_x w + w\partial_x) \partial_x^{-3} (\partial_x w + w\partial_x), \\ H_{-3} &= -\frac{1}{2} \int dx \, (16w_{xx}^2 w^{-\frac{7}{2}} - 35w_x^4 w^{-\frac{11}{2}}). \end{aligned}$$

Using the Hamiltonian operators P_1 and P_2 , it is possible to construct the negative as well as positive hierarchies of flows. For the negative hierarchy, the first three flows are

$$\begin{aligned} w_{t_1} &= P_1 \frac{\delta H_0}{\delta w} = P_2 \frac{\delta H_{-1}}{\delta w} = P_3 \frac{\delta H_{-2}}{\delta w} = 0, \\ w_{t_2} &= P_1 \frac{\delta H_1}{\delta w} = P_2 \frac{\delta H_0}{\delta w} = w_x, \end{aligned} \quad (5)$$

$$w_{t_3} = P_1 \frac{\delta H_2}{\delta w} = P_2 \frac{\delta H_1}{\delta w} = P_3 \frac{\delta H_0}{\delta w} = -w_x(\partial_x^{-2} w) - 2w(\partial_x^{-1} w), \quad (6)$$

where

$$H_0 = \int dx w, \quad H_1 = \frac{1}{2} \int dx (\partial_x^{-1} w)^2, \quad H_2 = \frac{1}{2} \int dx (\partial_x^{-2} w)(\partial_x^{-1} w)^2. \quad (7)$$

The equation (6) is the HS equation (1) after identifying $t_3 = t, w = u_{xx}$. Notice that w_{t_2} is not a tri-Hamiltonian because

$$P_3^{-1} w_x = \frac{1}{4} w^{-\frac{1}{2}} \partial_x^{-1} w^{-\frac{1}{2}} \partial_x^3 w^{-\frac{1}{2}} \partial_x^{-1} w^{-\frac{1}{2}} w_x = 0.$$

The HS equation (1) is connected with the Liouville equation by a reciprocal transformation. Indeed let us rewrite this equation in the conservative form as

$$v_t = (v \partial^{-2} v^2)_x$$

where $v^2 = u$. This form implies the reciprocal transformation as

$$dy = v dx + (v \partial^{-2} v^2) dt, \quad d\tau = dt$$

and therefore

$$\partial_x = v \partial_y, \quad \partial_t = \partial_\tau + (v \partial_x^{-2} v^2) \partial_y.$$

It is straightforward to show that the resulted equation reads as

$$(\log v)_{y\tau} = v,$$

which is the celebrated the Liouville equation.

3 Even Supersymmetric Hunter - Saxton equation

The even supersymmetric HD equation has been constructed by the supersymmetrization of two Hamiltonian operators P_1 and P_2 in [2]. The supersymmetric partner of the second Hamiltonian operator P_2 corresponds in fact to the supersymmetric centerless Virasoro algebra. The supersymmetric Hamiltonian structures are given by

$$\begin{aligned} \hat{K}_1 &= \mathcal{D} \partial_x^2, \\ \hat{K}_2 &= \frac{1}{2} [W \partial_x + 2 \partial_x W + (\mathcal{D} W) \mathcal{D}], \end{aligned}$$

where $\mathcal{D} = \partial_\theta + \theta \partial_x$ and $W = \chi(x, t) + \theta u(x, t)$ is a fermionic super field.

In these variables the even supersymmetric HD equation is

$$\begin{aligned} W_t &= \hat{K}_1 \frac{\delta H_{-1}}{\delta w} = \hat{K}_2 \frac{\delta H_{-2}}{\delta W} \\ &= \frac{1}{4} \partial_x^2 \left[-4 W_x (\mathcal{D} W)^{-\frac{3}{2}} + 3 W (\mathcal{D} W_x) (\mathcal{D} W)^{-\frac{5}{2}} \right], \end{aligned} \quad (8)$$

where

$$\begin{aligned} H_{-1} &= \int dx d\theta W (\mathcal{D} W)^{-\frac{1}{2}} = \frac{1}{2} \int dx \left(2 u^{-\frac{1}{2}} - \chi_x \chi u^{-\frac{3}{2}} \right), \\ H_{-2} &= \frac{1}{16} \int dx d\theta \left(W_x (\mathcal{D} W_x) (\mathcal{D} W)^{-\frac{5}{2}} - 15 W W_x W_{xx} (\mathcal{D} W)^{-\frac{7}{2}} \right) \\ &= \frac{1}{16} \int dx \left(u_x^2 u^{-\frac{5}{2}} + 16 \chi_{xx} \chi_x u^{-\frac{5}{2}} - 15 \chi_{xx} \chi u_x u^{-\frac{7}{2}} + 15 \chi_x \chi u_{xx} u^{-\frac{7}{2}} \right). \end{aligned}$$

In components the equation (8) reads

$$\begin{aligned}\chi_t &= -\frac{1}{2}\left((\chi u^{-\frac{3}{2}})_x + \chi_x u^{-\frac{3}{2}}\right)_{xx}, \\ u_t &= \frac{1}{4}\left(2u^{-\frac{1}{2}} + 3\chi_x \chi u^{-\frac{5}{2}}\right)_{xxx}.\end{aligned}$$

In fact this supersymmetric generalization constitutes the tri-Hamiltonian system also, i.e.

$$W_t = \hat{K}_3 \frac{\delta H_{-3}}{\delta W} = \hat{K}_2 \hat{K}_1^{-1} \hat{K}_2 \frac{\delta H_{-3}}{\delta W}, \quad (9)$$

where

$$\begin{aligned}H_{-3} &= -\frac{1}{384} \int dx d\theta W \left(420 W_{xxx} W_{xx} (\mathcal{D}W)^{-\frac{9}{2}} + 2394 W_{xx} W_x (\mathcal{D}W_{xx}) (\mathcal{D}W)^{-\frac{11}{2}} \right. \\ &\quad \left. - 875 (\mathcal{D}W_{xxx}) (\mathcal{D}W_x) (\mathcal{D}W)^{-\frac{9}{2}} + 7 (\mathcal{D}W_{xx})^2 (\mathcal{D}W)^{-\frac{9}{2}} + 348 (\mathcal{D}W_{4x}) (\mathcal{D}W)^{-\frac{7}{2}} \right).\end{aligned}$$

The proof that \hat{K}_3 satisfies the Jacobi identity is postponed to the appendix.

The negative hierarchy of the supersymmetric HD equation [2] is defined as in the classical case. It reads as

$$\begin{aligned}W_{t_1} &= \hat{K}_1 \frac{\delta H_0}{\delta W} = \hat{K}_2 \frac{\delta H_{-1}}{\delta W} = \hat{K}_3 \frac{\delta H_{-2}}{\delta W} = 0, \\ W_{t_2} &= \hat{K}_1 \frac{\delta H_1}{\delta W} = \hat{K}_2 \frac{\delta H_0}{\delta W} = W_x, \\ W_{t_3} &= \hat{K}_1 \frac{\delta H_2}{\delta W} = \hat{K}_2 \frac{\delta H_1}{\delta W} = \hat{K}_3 \frac{\delta H_0}{\delta W}\end{aligned} \quad (10)$$

$$= -\frac{3}{2} W (\mathcal{D}^{-1} W) - W_x (\mathcal{D}^{-3} W) - \frac{1}{2} (\mathcal{D}W) (\partial_x^{-1} W), \quad (11)$$

where

$$\begin{aligned}H_0 &= \int dx d\theta W, \quad H_1 = -\frac{1}{4} \int dx d\theta (\mathcal{D}^{-3} W) W, \\ H_2 &= \frac{1}{2} \int dx d\theta (\mathcal{D}^{-1} W) (\partial_x^{-1} W) (\mathcal{D}^{-3} W).\end{aligned}$$

We remark here that the equation (11) is the even supersymmetric extension of the HS equation which has been considered first time in [2] and later rediscovered in [10].

This supersymmetric equation could be rewritten in the new bosonic super field $U = -(\mathcal{D}W)/2$ as

$$U_{t_3} = 2U_x (\partial_x^{-2} U) + 4U (\partial_x^{-1} U) - (\mathcal{D}U) (\mathcal{D}^{-3} U), \quad (12)$$

which has a conservation law

$$\left(U^{\frac{1}{4}} \right)_{t_3} = \mathcal{D} \left(U^{\frac{1}{4}} (\mathcal{D}^{-3} U) + 2 (\mathcal{D}U^{\frac{1}{4}}) (\partial_x^{-2} U) \right). \quad (13)$$

In that way we prove that

$$H = \int dx d\theta U^{\frac{1}{4}} = \frac{1}{4} \int dx \xi q^{-\frac{3}{4}}$$

is a fermionic conserved quantity with the non classical analog, where $U = q + \theta\xi$.

To construct a reciprocal transformation for the equation (12), we turn to the Proposition 1 established in [11]. Thus, in addition to the conservation law (13) a potential is required. In this case, we have

$$\mathcal{D}\left(2U^{\frac{1}{2}}(\partial_x^{-2}U)\right) = 2U^{\frac{1}{4}}\left(U^{\frac{1}{4}}(\mathcal{D}^{-3}U) + 2(\mathcal{D}U^{\frac{1}{4}})(\partial_x^{-2}U)\right).$$

Hence, a reciprocal transformation is formulated as

$$\mathcal{D} = U^{\frac{1}{4}}\mathbb{D}, \quad (14)$$

$$\frac{\partial}{\partial t_3} = \frac{\partial}{\partial \tau} + 2U^{\frac{1}{2}}(\partial_x^{-2}U)\frac{\partial}{\partial y} + \left(U^{\frac{1}{4}}(\mathcal{D}^{-3}U) + 2(\mathcal{D}U^{\frac{1}{4}})(\partial_x^{-2}U)\right)\mathbb{D}. \quad (15)$$

Applying the transformation (14)-(15) we obtain a new supersymmetric generalization of the Liouville equation

$$(\log U)_{y\tau} = 4U^{\frac{1}{2}} - U^{-\frac{5}{4}}(\mathbb{D}U)(\mathbb{D}^{-1}U^{\frac{3}{4}}). \quad (16)$$

To see the connection with Liouville equation, we rewrite above equation in terms of components. It yields

$$(\log q)_{y\tau} = 4q^{\frac{1}{2}} - \frac{3}{4}\eta q^{-\frac{1}{4}}(\partial_x^{-1}\eta q^{\frac{3}{4}}), \quad (17)$$

$$\eta_{\tau} = 3\eta q^{\frac{1}{2}} - \frac{3}{4}q_x q^{-\frac{5}{4}}(\partial_x^{-1}\eta q^{\frac{3}{4}}), \quad (18)$$

where $\eta = q^{-1}\xi$. When $\eta = 0$, this system reduces to the Liouville equation.

4 Odd Supersymmetric Hunter - Saxton Equation

The other supersymmetric HD equation was worked out from a Lax operator and the associated Lax representation [2]. Indeed, the supersymmetric Lax operator

$$L = (\mathcal{D}W)^{-1}\mathcal{D}^4 - \frac{1}{2}W_x(\mathcal{D}W)^2\mathcal{D}^3 \quad (19)$$

leads to the following generalization of the Harry Dym equation

$$L_t = \left[L_{\geq 3}^{\frac{3}{2}}, L\right],$$

or

$$\begin{aligned} W_t = & \frac{1}{16}\left[8\mathcal{D}^5(\mathcal{D}W)^{-\frac{1}{2}} - 3\mathcal{D}(W_{xx}W_x(\mathcal{D}W)^{-\frac{5}{2}})\right. \\ & \left. + \frac{3}{4}(\mathcal{D}W_x)^2W_x(\mathcal{D}W)^{-\frac{7}{2}} - \frac{3}{4}\mathcal{D}^{-1}((\mathcal{D}W_x)^3(\mathcal{D}W)^{-\frac{7}{2}})\right], \end{aligned}$$

where W is a fermionic super field. For our next purposes let us rewrite this equation by means of $V = (\mathcal{D}W)^{\frac{1}{2}}$ as

$$V_t = -\frac{1}{8}\left[\partial_x^3(V^{-2}) - 3\mathcal{D}\partial_x((\mathcal{D}V)V_xV^{-4})\right]. \quad (20)$$

In the components the last equation becomes

$$\begin{aligned} u_t &= -\frac{1}{8}\partial_x\left(\partial_x^2(u^{-2}) - 3u_x^2u^{-4} - 3\chi_x\chi u^{-4}\right), \\ \chi_t &= \frac{1}{8}\partial_x^2\left(2\partial(u^{-3}\chi) - 3u_xu^{-4}\chi\right), \end{aligned}$$

where $V = u(x, t) + \theta\chi(x, t)$.

The classical HD equation (3) could be obtained in the bosonic sector where $\chi = 0$ and $u = w^{\frac{1}{2}}$. Thus both even and odd supersymmetric extensions contain the classical HD equation but their fermionic extensions are different.

The bi-Hamiltonian structure for (20) was constructed very recently in [11], it reads

$$V_t = \hat{P}_2 \frac{\delta \hat{H}_{-2}}{\delta V} = \hat{P}_3 \frac{\delta \hat{H}_{-3}}{\delta V}, \quad (21)$$

where

$$\hat{H}_{-2} = -\frac{1}{8} \int dx d\theta (\mathcal{D}V) V_x V^{-3} = -\frac{1}{8} \int dx (u_x^2 u^{-3} + \chi_x \chi u^{-3}), \quad (22)$$

$$\begin{aligned} \hat{H}_{-3} &= - \int dx d\theta (\mathcal{D}V) (4V_{xxx} V^{-5} - 30V_{xx} V_x V^{-6} + 35V_x^3 V^{-7}) \\ &= \int dx \left(4u_{xx}^2 u^{-5} - 15u_x^4 u^{-7} + 4\chi_{xx}\chi_x u^{-5} - \chi_x \chi (45u_x^2 u^{-7} - 20u_{xx} u^{-6}) \right), \end{aligned} \quad (23)$$

$$\hat{P}_2 = \mathcal{D}^3, \quad (24)$$

$$\begin{aligned} \hat{P}_3 &= V^{\frac{1}{2}} \mathcal{D} V^{-\frac{1}{2}} \mathcal{D} V \mathcal{D}^{-5} V \mathcal{D} V^{-\frac{1}{2}} \mathcal{D} V^{\frac{1}{2}} \\ &= \frac{1}{4} \left[(\mathcal{D}V) \partial_x^{-2} ((\mathcal{D}V) \mathcal{D} - V \partial_x - \partial_x V) + 2\partial_x V \partial_x^{-3} (2V \partial_x \mathcal{D} - 2(\mathcal{D}V_x) + 3\partial_x(\mathcal{D}V)) \right]. \end{aligned} \quad (25)$$

The above Hamiltonian operators in the components are given by

$$\hat{P}_2 = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x^2 \end{pmatrix}, \quad (26)$$

$$\hat{P}_3 = \frac{1}{4} \begin{pmatrix} \partial_x u \partial_x^{-3} u \partial_x + \chi \partial_x^{-2} \chi & -\chi \partial_x^{-1} (\partial_x^{-1} u \partial_x + u) + 2\partial_x u \partial_x^{-2} (-2\partial_x^{-1} \chi_x + 3\chi) \\ -(u + \partial_x u \partial_x^{-1}) \partial_x^{-1} \chi + 2(2\chi_x \partial_x^{-1} + 3\chi) \partial_x^{-2} u \partial_x & u^2 + u \partial_x^{-1} u \partial_x + \partial_x u \partial_x^{-1} u + \partial_x u \partial_x^{-2} u \partial_x + (\chi + 2\partial_x \chi \partial_x^{-1}) \partial_x^{-1} (2\partial_x^{-1} \chi \partial_x + \chi) \end{pmatrix}. \quad (27)$$

In passing, we notice that the Poisson brackets induced by \hat{P}_2 and \hat{P}_3 operators are given by

$$\{V(x, \theta), V(y, \theta')\} = \hat{P}_i \delta(x - y) (\theta - \theta'), \quad i = 2, 3. \quad (28)$$

As V is a bosonic super field and \hat{P}_i is an superfermionic operator hence we are dealing with the so called odd Poisson brackets.

The Jacobi identity for the operator P_3 is proved in [11] and in the appendix we present a quite different verification of this property.

In contrast to the classical case we do not know the first Hamiltonian operator for the odd supersymmetric HD equation (20). However we can construct the negative hierarchy using our bi - Hamiltonian structure only.

To this end, let us first find the supersymmetric analog of the classical Casimir functions for the equation (20). In the classical case $H_{-1} = \int dx u$ and $H_0 = \int dx u^2$ defined by (4), (7) are the Casimir functions for the Hamiltonians operators P_2 and P_1 respectively while H_0 is a Casimir function for P_3 .

In the supersymmetric case we find out that

$$\hat{H}_{-1} = \int dx d\theta V^{-1}(\mathcal{D}^{-1}V^2) = \int dx (u - 2\chi u^{-2}\partial_x^{-1}\chi u) \quad (29)$$

is conserved quantity for the equation (20) and is a Casimir function for \hat{P}_2 . On the other side, \hat{H}_{-2} defined by (22) is the Casimir function for \hat{P}_3 .

Interestingly we have an additional new Casimir function for the operator \hat{P}_2

$$\hat{G}_0 = \int dx d\theta V = \int dx \chi \quad (30)$$

whose parity is different from that of \hat{H}_{-1} (or \hat{H}_{-2}). If we use \hat{G}_0 to the construction of the negative hierarchy it appears that this hierarchy will be of a different parity than the odd supersymmetric HD equation (20) and from the physical point of view it may not be relevant. Indeed if we consider the following equation

$$V_\varsigma = P_3 \frac{\delta \hat{G}_0}{\delta V} = \frac{1}{4} [2\partial_x V \partial_x^{-2} \mathcal{D}V - (\mathcal{D}V) \partial_x^{-1} V], \quad (31)$$

which in the components reads

$$\begin{aligned} u_\varsigma &= \frac{1}{4} [-\chi \partial_x^{-1} u + 2\partial_x u \partial_x^{-2} \chi], \\ \chi_\varsigma &= \frac{1}{4} [u^2 + \chi \partial_x^{-1} \chi + \partial_x u \partial_x^{-1} u + 2\partial_x \chi \partial_x^{-2} \chi], \end{aligned}$$

then we have to assume that ς is a fermionic time.

From \hat{H}_{-1} and \hat{H}_{-2} we obtain the first member of the negative hierarchy of the odd supersymmetric HD equation as

$$V_{t_1} = \hat{P}_2 \frac{\delta \hat{H}_{-1}}{\delta V} = \hat{P}_3 \frac{\delta \hat{H}_{-2}}{\delta V} = 0. \quad (32)$$

In the classical case the second member of the negative hierarchy of HD equation is not tri - Hamiltonian but bi - Hamiltonian system generated by the operators P_1, P_2 and Hamiltonians H_1 and H_0 only. In order to construct the second member of the negative hierarchy in the supersymmetric case let us build the superfermionic partner of the classical $H_0 = \int dx w = \int dx u^2$ as in the equation (7). The result is

$$\hat{H}_0 = \int dx d\theta V(\mathcal{D}^{-1}V) = \int dx (u^2 + \chi(\partial^{-1}\chi)). \quad (33)$$

It is a conserved quantity for the odd supersymmetric HD equation (20) but not a Casimir function \hat{P}_2 or \hat{P}_3 . This Hamiltonian generates the following translation flow

$$V_{t_2} = \hat{P}_2 \frac{\delta \hat{H}_0}{\delta V} = V_x \quad (34)$$

which gives us in the bosonic limit the second negative flow of the classical HD hierarchy (10).

Now the conserved quantity H_0 generates the supersymmetric analog of the HS equation which is a bi - Hamiltonian system also

$$\begin{aligned} V_{t_3} &= \hat{P}_3 \frac{\delta \hat{H}_0}{\delta V} = \hat{P}_2 \frac{\delta \hat{H}_1}{\delta V} \\ &= \partial_x \left(V \mathcal{D}^{-3} [V(\mathcal{D}^{-1}V)] \right) - \frac{1}{2} (\mathcal{D}V) \partial_x^{-1} [V(\mathcal{D}^{-1}V)], \end{aligned} \quad (35)$$

where

$$\begin{aligned} \hat{H}_1 &= \int dx d\theta \left[2(\mathcal{D}^{-1}V)V\mathcal{D}^{-3} - (\mathcal{D}V)(\mathcal{D}^{-3}V)\partial^{-1} \right] V\mathcal{D}^{-1}V \\ &= \int dx \left[2u^2\partial^{-2} + 2\chi(\partial^{-1}\chi)\partial^{-2} - \chi(\partial^{-2}\chi)\partial^{-1} \right] (u^2 + \chi\partial^{-1}\chi) - \\ &\quad 2(\partial^{-1}\chi)u\partial^{-1}u\partial^{-1}\chi + (\partial^{-2}\chi)u^2\partial^{-1}\chi. \end{aligned}$$

In components we have

$$\begin{aligned} u_{t_3} &= \partial_x u \partial_x^{-2} (u^2 + \chi \partial_x^{-1} \chi) - \frac{1}{2} \chi \partial_x^{-1} u \partial_x^{-1} \chi, \\ \chi_{t_3} &= \chi_x \partial_x^{-2} (u^2 + \chi \partial_x^{-1} \chi) + \frac{3}{2} \chi \partial_x^{-1} (u^2 + \chi \partial_x^{-1} \chi) + \frac{1}{2} u_x \partial_x^{-1} u \partial_x^{-1} \chi + u^2 \partial_x^{-1} \chi. \end{aligned}$$

If we take $\chi = 0$ and $w = u^2$, then w_{t_3} is nothing but the HS equation.

Now we turn to the construction of possible reciprocal transformation for the supersymmetric HS equation (35). It is observed that the equation (35) has the following conservation law

$$\left(V^{\frac{1}{2}} \right)_{t_3} = \frac{1}{2} \frac{\partial}{\partial x} \left(V^{\frac{1}{2}} \mathcal{D}^{-3} [V(\mathcal{D}^{-1}V)] \right) + \frac{1}{4} \mathcal{D} \left(V^{-\frac{1}{2}} (\mathcal{D}V) \mathcal{D}^{-3} [V(\mathcal{D}^{-1}V)] \right). \quad (36)$$

Furthermore, a potential can be introduced for the product of the conserved density and flux in the above conservation law

$$\mathcal{D} \left(V \mathcal{D}^{-3} [V(\mathcal{D}^{-1}V)] \right) = V^{\frac{1}{2}} \mathcal{D} \left(V^{\frac{1}{2}} \mathcal{D}^{-3} [V(\mathcal{D}^{-1}V)] \right) + \frac{1}{2} (\mathcal{D}V) \mathcal{D}^{-3} [V(\mathcal{D}^{-1}V)].$$

Hence, we have a reciprocal transformation given by

$$\mathcal{D} = V^{\frac{1}{2}} \mathbb{D}, \quad (37)$$

$$\begin{aligned} \frac{\partial}{\partial t_3} &= \frac{\partial}{\partial \tau} + V \mathcal{D}^{-3} [V(\mathcal{D}^{-1}V)] \frac{\partial}{\partial y} \\ &\quad + \frac{1}{2} \left(V^{-\frac{1}{2}} (\mathcal{D}V) \mathcal{D}^{-3} [V(\mathcal{D}^{-1}V)] + V^{\frac{1}{2}} \partial_x^{-1} [V(\mathcal{D}^{-1}V)] \right) \mathbb{D}, \end{aligned} \quad (38)$$

which could be applied to the supersymmetric HS equation (35). A direct calculation gives us

$$V_\tau = V\mathbb{D}^{-1}[V^{\frac{1}{2}}(\mathbb{D}^{-1}V^{\frac{1}{2}})]. \quad (39)$$

By introducing $V = (\mathbb{D}\Phi)^2$, the last equation is rewritten in a neat form

$$2\frac{\partial}{\partial\tau}\mathbb{D}\log(\mathbb{D}\Phi) = (\mathbb{D}\Phi)\Phi \quad (40)$$

which is a new supersymmetric generalization of the Liouville equation. To see it, we just rewrite this equation in components $\Phi = \xi + \varrho u$ as

$$\begin{aligned} 2(\log u)_{y\tau} &= u^2 + \xi_y\xi, \\ 2(u^{-1}\xi_y)_\tau &= u\xi, \end{aligned}$$

when $\xi = 0$ this system goes back to the Liouville equation.

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Appendix

In order to prove that the operator $\hat{K}_3 = \hat{K}_2\hat{K}_1^{-1}\hat{K}_2$ satisfies the Jacobi identity we use the idea of the decompression of the Hamiltonian operators described in [14]. To this end let us embed the \hat{K}_3 operator into the two dimensional matrix as

$$J = \begin{pmatrix} 2V_x + 3V\partial + (\mathcal{D}V)\mathcal{D} & \hat{K}_2 \\ \hat{K}_2 & \hat{K}_1 + 2V_x + 3V\partial + (\mathcal{D}V)\mathcal{D} \end{pmatrix}$$

where V is a new superfermionic function.

This new J operator does not contain the nonlocal terms hence it is easy to prove that the Jacobi identity holds for it indeed. Now using the Dirac reduction procedure one can easily recognize that in the case $V = 0$ the operator J reduces to \hat{K}_3 , hence the last operator satisfies the Jacobi identity also.

For the odd operator \hat{P}_3 , we reformulate it as

$$\hat{P}_3 = V^{\frac{1}{2}}\mathcal{D}V^{-\frac{1}{2}}\mathcal{D}V\mathcal{D}^{-5}V\mathcal{D}V^{-\frac{1}{2}}\mathcal{D}V^{\frac{1}{2}} = -Q\mathcal{D}^{-5}Q^\star,$$

where

$$Q = V^{\frac{1}{2}}\mathcal{D}V^{-\frac{1}{2}}\mathcal{D}V.$$

Similarly to the previous case let us embed \hat{P}_3 operator to the graded two dimensional matrix

$$\hat{J} = \begin{pmatrix} 0 & -\frac{1}{2}Q \\ \frac{1}{2}Q^\star & \mathcal{D}^5 - 2W_x - 3W\partial - (\mathcal{D}W)\mathcal{D} \end{pmatrix}$$

where W is a new superfermionic function.

This matrix generates the following Poisson brackets for the fields U_i

$$\{U_i(x, \theta), U_j(y, \theta')\} = \hat{J}(i, j)\delta(x - y)(\theta - \theta'), \quad i, j = 1, 2$$

with identifications $U_1 = V, U_2 = W$. Remember that the test functions are graded vectors it is easy to verify that \hat{J} does satisfy the Jacobi identity. Applying the Dirac reduction procedure with $W = 0$, we find that the reduced operator is just $\frac{1}{4}\hat{P}_3$, which satisfies the Jacobi identity too.

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